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# Lifted Transformations on the Tangent Bundle, and Symmetries of Particle Motion

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We define affine transport lifts on the tangent bundle by associating a transport rule for tangent vectors with a vector field on the base manifold. Our aim is to develop tools for the study of kinetic/dynamic symmetries in particle motion. The new lift unifies and generalizes all the various existing lifted vector fields, with clear geometric interpretations. In particular, this includes the important but little-known "matter symmetries" of relativistic kinetic theory. We find the affine dynamical symmetries of general relativistic charged particle motion, and we compare this to previous results and to the alternative concept of "matter symmetry."

## 1. INTRODUCTION

Vector fields on the tangent bundle $TM$, arising as the lifts of vectors or of transformations on the base space $M$, have been defined and applied in differential geometry, Lagrangian mechanics, and relativity; for example, the complete (natural or Lie), horizontal and vertical lifts (Yano and Ishihara, 1973; Crampin, 1983; Prince and Crampin, 1984; Crampin and Pirani, 1986), the projective and conformal lifts of Iwai (1977), and the matter symmetries of Berezdivin and Sachs (1973) (Oliver and Davis, 1979).

Our aim is to find a more general way of lifting from $M$ to $TM$ than the usual definitions that involve only the vector field on $M$, and possibly the connection on $M$. In fact, the matter symmetries of Berezdivin and Sachs (1973) are a step in this direction. We generalize their concept in a way that gives a clear geometric foundation to all the lifts previously defined, and to new lifts which can be defined. While the differences among the known lifts are as important as their common features, a unified approach can give new

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insights into kinetic and dynamic symmetries of particle motion. Our main interest is in applications to general relativistic kinetic theory. However, the concepts and formalism are more widely applicable.

The main idea is to associate a transport rule for tangent vectors with a vector field on $M$. This defines a vector field on $TM$—the transport lift. The class of *affine transport lifts* (ATLs) generalizes all previously defined lifts in a unified and geometrical way. We find conditions under which ATLs are dynamical symmetries for particle trajectories in (semi-) Riemannian manifolds, thus throwing new light on earlier results.

In Section 2 we give a brief and simplified summary of the relevant differential geometry of the manifold and tangent bundle. In Section 3 we define the transport rule for tangent vectors on $M$ and the corresponding definition of a transport lift on $TM$. Affine transport lifts are then defined and their properties are discussed. Previous lifts are geometrically interpreted as special cases of ATLs. Particular attention is focused on matter symmetries. In Section 4 we find the conditions under which ATLs are dynamical symmetries, generalizing previous results. The special case of dynamical matter symmetries is also discussed.

## 2. LOCAL GEOMETRY OF THE TANGENT BUNDLE

We give a brief summary of the relevant local differential geometry of the tangent bundle assuming only a knowledge of basic tensor analysis on manifolds. For further details, see, for example, Crampin and Pirani (1986) or Yano and Ishihara (1973). Consider a (semi-) Riemannian $n$-manifold $(M, g)$ with local coordinates $x^a$ and metric connection (Christoffel symbols)

$$\Gamma^a{}_{bc}=\tfrac{1}{2}g^{ad}(g_{db,c}+g_{cd,b}-g_{bc,d})$$

The tangent bundle $TM$ is the union of all tangent spaces (fibers) $T_xM$, $x\in M$. In relativistic kinetic theory (RKT) the phase space arises out of $TM$ by restriction to future-directed, nonspacelike tangent vectors (Maartens and Maharaj, 1985).

Local coordinates $x^a$ on $M$ induce local coordinates $\xi^I=(x^a,p^b)$ on $TM$, where $p^a$ are the coordinate components of the vector $p$:

$$p=p^a\frac{\partial}{\partial x^a}$$

More precisely, $\xi^I=(x^a\circ\pi,\ dx^b\circ v)$, where $\pi$ is the projection to the base manifold $[\pi(\xi)=x]$ and $v(\xi)=p$. Coordinate transformations $x^a\rightarrow x^{a'}(x^a)$

induce the transformation

$$\xi^I \to \xi^{I'} = \left( x^{a'}(x^a), \frac{\partial x^{b'}}{\partial x^b} p^b \right) \tag{1}$$

on $TM$.

Any smooth vector field $\Sigma = d/d\sigma$ on $TM$ can be expressed locally as

$$\Sigma = \Sigma^I \frac{\partial}{\partial \xi^I} = \alpha^a(x, p) \frac{\partial}{\partial x^a} + \beta^a(x, p) \frac{\partial}{\partial p^a} \tag{2a}$$

where

$$\alpha^a(x, p) = \frac{dx^a}{d\sigma}, \qquad \beta^a(x, p) = \frac{dp^a}{d\sigma} \tag{2b}$$

give the integral curves $\xi^I(\sigma)$ of $\Sigma$.

The coordinate transformation (1) induces the following transformation of the basis vector fields $\partial/\partial\xi^I$:

$$\frac{\partial}{\partial x^a} = \frac{\partial x^{a'}}{\partial x^a} \frac{\partial}{\partial x^{a'}} + \frac{\partial^2 x^{a'}}{\partial x^a \, \partial x^b} p^b \frac{\partial}{\partial p^{a'}} \tag{3a}$$

$$\frac{\partial}{\partial p^a} = \frac{\partial x^{a'}}{\partial x^a} \frac{\partial}{\partial p^{a'}} \tag{3b}$$

which implies

$$\alpha^a \to \alpha^{a'} = \frac{\partial x^{a'}}{\partial x^a} \alpha^a \tag{4a}$$

$$\beta^a \to \beta^{a'} = \frac{\partial x^{a'}}{\partial x^a} \beta^a + \frac{\partial^2 x^{a'}}{\partial x^a \, \partial x^b} p^b \alpha^a \tag{4b}$$

Thus, the $\beta^a$ do not transform like vector components on $M$, i.e., the components $\Sigma^I = (\alpha^a, \beta^b)$ in the basis $\{\partial/\partial x^a, \partial/\partial p^b\}$ are not a covariant splitting of $\Sigma$. Using (3), (4), and the $\Gamma^a{}_{bc}$ transformation law, we arrive at a covariant splitting of vector components via the anholonomic "connection basis" $\{H_a, V_b\}$ of horizontal and vertical vector fields (Crampin and Pirani, 1986):

$$H_a = \frac{\partial}{\partial x^a} - \Gamma^b{}_{ca} p^c \frac{\partial}{\partial p^b}, \qquad V_a = \frac{\partial}{\partial p^a} \tag{5}$$

By (3) and the transformation law for the connection, these obey the transformation law

$$H_{a'}=\frac{\partial x^a}{\partial x^{a'}}H_a, \qquad V_{a'}=\frac{\partial x^a}{\partial x^{a'}}V_a \tag{6}$$

Then we find that

$$\Sigma=b^a(x,p)H_a+c^a(x,p)V_a=b^{a'}(x',p')H_{a'}+c^{a'}(x',p')V_{a'}$$

where, by (1) and (6),

$$b^a=\frac{\partial x^a}{\partial x^{a'}}b^{a'}, \qquad c^a=\frac{\partial x^a}{\partial x^{a'}}c^{a'}$$

Thus, the connection basis vectors and the components of vector fields in this basis all transform like vector fields on $M$. The Lie brackets of the basis vectors are

$$[V_a, V_b]=0 \tag{7}$$

$$[H_a, V_b]=\Gamma^c{}_{ab}V_c \tag{8}$$

$$[H_a, H_b]=-R^d{}_{cab}p^cV_d \tag{9}$$

The vector field

$$\Gamma=p^aH_a \tag{10a}$$

has, by (2) and (5), integral curves $\xi^I(v)$ on $TM$ which are the natural lifts of geodesics $x^a(v)$ on $M$:

$$v\to\xi^I(v)=(x^a(v),\dot{x}^b(v)), \qquad \dot{x}^a\equiv dx^a/dv$$

$$\frac{D^2x^a}{dv^2}\equiv\ddot{x}^a+\Gamma^a{}_{bc}\dot{x}^b\dot{x}^c=0$$

Then $\Gamma$ is called the geodesic spray. In RKT, $\Gamma$ is the Liouville vector field, since the Liouville (Vlasov) equation is

$$\Gamma f=0 \tag{11}$$

where $f=f(x,p)$ is the distribution function for uncharged particles (Maartens and Maharaj, 1985).

In the case of charged particles (charge $e$), the Liouville vector field generalizes to

$$\Gamma_{(e)}=\Gamma+eF^a{}_bp^bV_a \tag{10b}$$

where $F$ is the electromagnetic field tensor. This follows since, by (2), (5), and (10), the integral curves of $\Gamma_{(e)}$ are the lifts of charged particle trajectories (Stephani, 1982):

$$p^a = \frac{dx^a}{dv}, \qquad \frac{D^2 x^a}{dv^2} = eF^a{}_b p^b$$

where $v = $ (proper time)/(rest mass). Then the Liouville equation (11) generalizes to the charged particle case:

$$\Gamma_{(e)} f = 0$$

Note that $\Gamma_{(e)}$, and in particular $\Gamma = \Gamma_{(0)}$, is always tangent to the hypersurfaces defined by

$$g_{ab}(x) p^a p^b = -m^2$$

This follows from a general result for scalars on $TM$ that are defined by symmetric tensors on $M$: for

$$q = Q_{a_1 a_2 \ldots a_r}(x) p^{a_1} p^{a_2} \ldots p^{a_r}$$

we have, by (10) (with parentheses denoting symmetrization)

$$\Gamma_{(e)} q = Q_{(a_1 a_2 \ldots a_r ; b)} p^{a_1} p^{a_2} \ldots p^{a_r} p^b + er F^b_{(a_1} Q_{a_2 \ldots a_r) b} p^{a_1} p^{a_2} \ldots p^{a_r} \tag{12}$$

which reduces to the well-known result when $e = 0$. Thus, in particular, (12) implies

$$\Gamma_{(e)} m^2 = 0$$

since $g_{ab;c} = 0 = F_{(ab)}$. Also, from (12), we get

$$\Gamma q = 0 \Leftrightarrow Q_{a_1 a_2 \ldots a_r} \quad \text{is a rank-}r \text{ Killing tensor}$$

In particular, Killing vector fields define linear first integrals of geodesic motion:

$$y = Y_a(x) p^a, \qquad Y_{(a;b)} = 0 \Rightarrow \Gamma y = 0$$

A Newtonian dynamical system (Iwai, 1977) on a Riemannian manifold is defined by a force vector field $f^a$, which determines the Newtonian trajectories via

$$\frac{Dp^a}{dt} = f^a(x)$$

The natural lifts $t \to (x^a(t), \dot{x}^b(t))$ define the Newtonian dynamical vector field on $TM$:

$$\Gamma = p^a H_a + f^a V_a \tag{13}$$

In fact this dynamical field also applies to relativistic motion under a velocity-independent four-force $f^a$.

For a vector field

$$Y = Y^a(x) \frac{\partial}{\partial x^a}$$

on $M$, various lifted vector fields have been defined on $TM$:

(i) Horizontal lift:

$$Y \to \bar{Y} = Y^a(x) H_a \tag{14}$$

(ii) Vertical lift:

$$Y \to \hat{Y} = Y^a(x) V_a \tag{15}$$

(iii) Complete lift:

$$Y \to \tilde{Y} = Y^a(x) H_a + Y^a{}_{;b}(x) p^b V_a \tag{16}$$

(iv) Iwai's lift:

$$Y \to Y^\dagger = \tilde{Y} - 2\psi(x) p^a V_a \tag{17}$$

Here $\psi$ is proportional to $Y^a{}_{;a}$ in (17). We shall give a geometric explanation of these lifts in the next section.

We can also define the vertical lift of a rank-2 tensor field (Yano and Ishihara, 1973):

$$A \to \hat{A} = A^a{}_b(x) p^b V_a \tag{18}$$

with a special case being the Euler vector field (Crampin and Pirani, 1986)

$$\Delta = \hat{\delta} = p^a V_a \tag{19}$$

Matter symmetries in RKT have been defined by Berezdivin and Sachs (1973) in terms of a vector field $Y$ and a skew rank-2 tensor field $A$ on $M$:

$$(Y, A) \to Y^a(x) H_a + A^a{}_b(x) p^b V_a, \qquad A_{(ab)} = 0 \tag{20}$$

We shall explain the meaning of (20) in the next section.

To end this brief summary, we recall the definition of a dynamical symmetry (Crampin, 1983; Prince and Crampin, 1984). A dynamical system on $M$ is defined by a congruence of trajectories on $TM$. The tangent vector field to these trajectories is the dynamical vector field $\Gamma$. In the case of geodesic trajectories (free-fall particle motion), $\Gamma$ is given by (10a); for relativistic charged particle motion, by (10b); and for a Newtonian system, by (13).

A dynamical symmetry is a vector field $\Sigma$ that maps trajectories into trajectories with possibly rescaled tangent vector field. Thus $(\exp \varepsilon \mathscr{L}_\Sigma)\Gamma$ is parallel to $\Gamma$, and so

$$\mathscr{L}_\Sigma \Gamma \equiv [\Sigma, \Gamma] = -\psi \Gamma \tag{21}$$

for some $\psi(x,p)$, is the condition for $\Sigma$ to be a dynamical symmetry. The nature of the rescaling depends on $\psi(x,p)$. If $\psi = \psi(x)$, then the rescaling is constant on each fiber, $T_x M$. If $\psi = 0$, then there is no rescaling and $\Sigma$ is said to be a Lie symmetry on $TM$.

## 3. TRANSPORT LIFTS

Let $Y = d/d\sigma$ be a vector field on $M$ and $\Lambda$ a smooth local rule governing the transport of tangent vectors along the integral curves of $Y$. Thus any $u^a$ at $x^a(\sigma)$ is mapped under $\Lambda$ to $u'^a$ at $x'^a = x^a(\sigma + \varepsilon)$:

$$u'^a = \Lambda^a(x, u;\ \varepsilon)$$

This defines curves $(x^a(\sigma), p^b(\sigma))$ in $TM$, with

$$\frac{dx^a}{d\sigma} = Y^a(x), \quad \frac{dp^a}{d\sigma} \equiv \lambda^a(x, p) \tag{22a}$$

where the generator of $\Lambda$ is (Eisenhart, 1961)

$$\lambda^a(x, p) = [\partial \Lambda^a(x, p;\ \varepsilon)/\partial \varepsilon]_{\varepsilon=0} \tag{22b}$$

We can define a vector field on $TM$ with integral curves $(x^a(\sigma), p^b(\sigma))$ given by (22). We call this the *transport lift* on $TM$ of the vector field $Y$ and of the transport rule $\Lambda$ along $Y$. By (2), the transport lift is locally given by

$$(Y, \Lambda) \to Y^a(x) \frac{\partial}{\partial x^a} + \lambda^a(x, p) \frac{\partial}{\partial p^a} \tag{23}$$

where $\lambda^a(x, p)$ is defined by (22b). [By (4b), $\lambda^a$ does not transform covariantly, unless $Y = 0$.] Note that (23) preserves the fibers of $TM$, i.e., under the flow of (23), $T_x M$ is mapped to $T_{x'} M$. This property is also reflected in the following. If $\phi_\varepsilon$ is the flow of $Y$ on $M$ and $\Phi_\varepsilon$ is the flow of (23) on $TM$, then

$$\phi_\varepsilon \circ \pi = \pi \circ \Phi_\varepsilon$$

The transport lift (23) combines the point transformations generated by $Y$ on $M$ with the tangent vector transformations generated by $\Lambda$ on $M$. In fact, the transformations generated by (23) on $TM$ are

$$\Phi_\varepsilon: \quad (x, p) \to ([\exp\, \varepsilon Y]x, [\exp\, \varepsilon\lambda]p) \tag{24}$$

A covariant splitting of (23) is given by the connection basis (5):

$$(Y, \Lambda) \to Y^a(x)H_a + [\lambda^a(x, p) + \Gamma^a{}_{bc}(x)p^b Y^c(x)]V_a$$

[Note that by (4b), the vertical component transforms covariantly.] This covariant form makes it clear that, in general, the transport rule $\Lambda$ along $Y$ is not defined purely by tensor fields on $M$. However, this is the case for an affine transport rule, for which

$$\Lambda^a(x, u;\, \varepsilon) = \Omega^a{}_b(x;\, \varepsilon)u^b + K^a(x;\, \varepsilon)$$

Thus, the *affine transport lift* (ATL) of $(Y, \Lambda)$ on $M$ has the form

$$Y^{(A,k)} = Y^a(x)H_a + [A^a{}_b(x)p^b + k^a(x)]V_a \tag{25a}$$

where

$$A^a{}_b(x) = \omega^a{}_b(x) + \Gamma^a{}_{bc}(x)Y^c(x) \tag{25b}$$

and

$$\omega^a{}_b(x) = [\partial\Omega^a{}_b(x;\, \varepsilon)/\partial\varepsilon]_{\varepsilon=0} \tag{25c}$$

$$k^a(x) = [\partial K^a(x;\, \varepsilon)/\partial\varepsilon]_{\varepsilon=0} \tag{25d}$$

In this case we have

$$\lambda^a(x, p) = \omega^a{}_b(x)p^b + k^a(x)$$

so that by (4b) we get the transformation laws

$$k^{a'} = \frac{\partial x^{a'}}{\partial x^a}\, k^a$$

$$\omega^{a'}{}_{b'} = \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^b}{\partial x^{b'}}\, \omega^a{}_b + \frac{\partial x^b}{\partial x^{b'}}\frac{\partial^2 x^{a'}}{\partial x^a\, \partial x^b}\, Y^a$$

It follows that $k$ is a vector field on $M$, whereas $\omega$ is not a tensor field unless $Y=0$. Furthermore, $A$ as defined by (25b) is a tensor field, and the vertical component in (25a) therefore transforms covariantly, as it must. The transport rule $\Lambda$ is thus covariantly determined by $A$ and $k$. Note that by (25b), $A$ is not independent of $Y$.

By (25) and (2), the integral curves $\xi^I(\sigma)$ of $Y^{(A,k)}$ satisfy

$$\frac{dx^a}{d\sigma} = Y^a(x) \tag{26a}$$

$$\frac{dp^a}{d\sigma} = \omega^a{}_b(x)p^b + k^a(x) = [A^a{}_b(x) - \Gamma^a{}_{bc}(x)\, Y^c(x)]p^b + k^a(x) \tag{26b}$$

We can rewrite (26b) as

$$\frac{Dp^a}{d\sigma} = A^a{}_b p^b + k^a$$

which shows that $A$ and $k$ determine the rate of change of tangent vectors under $\Lambda$ relative to parallel transport. In the case $k=0$, we get a particularly simple interpretation of $A$:

$$A^a{}_b u^b = \nabla_Y u^a \qquad \text{or} \qquad A(u) = \nabla_Y u \tag{27}$$

for all $u$ along $Y$. This equation is important for the geometric construction of lifts (see below).

Note that by (14), (15), (18), and (25) we can write the ATL as

$$(Y, \Lambda) \to Y^{(A,k)} = \bar{Y} + \hat{A} + \hat{k}$$

However, this obscures the fact that $A$ is tied to $Y$ via (25b). The class of *linear transport lifts* (LTLs) arises as the special case $k^a = 0$, and we write

$$Y^{(A)} \equiv Y^{(A,0)}$$

LTLs encompass all previously defined lifts apart from the vertical lift (15).

Now from (25a) we get

$$\alpha\, Y^{(A,k)} + \beta Z^{(B,l)} = (\alpha\, Y + \beta Z)^{(\alpha A + \beta B, \alpha k + \beta l)} \tag{28}$$

for any scalars $\alpha$, $\beta$ on $M$. (Note again that $A$ and $B$ depend, respectively, on $Y$ and $Z$. In particular, this means that in general the taking of the affine transport lift is not a linear process.) Thus the ATLs form a linear subspace.

Furthermore, (7)–(9) give

$$\begin{aligned}[Y^{(A,k)}, Z^{(B,l)}] = [Y, Z]^a H_a &+ (B^a{}_{b;c} Y^c - A^a{}_{b;c} Z^c \\ &+ B^a{}_c A^c{}_b - A^a{}_c B^c{}_b - R^a{}_{bcd}\, Y^c Z^d) p^b V_a \\ &+ (l^a{}_{;b}\, Y^b - k^a{}_{;b} Z^b - A^a{}_b l^b + B^a{}_b k^b)\, V_a\end{aligned}$$

which may be rewritten as

$$[Y^{(A,k)}, Z^{(B,l)}] = [Y, Z]^{(C,m)} \tag{29a}$$

where

$$C = \nabla_Y B - \nabla_Z A - [A, B] - R(Y, Z) \quad \text{(29b)}$$

$$m = \nabla_Y l - \nabla_Z k - A(l) + B(k) \quad \text{(29c)}$$

$C$ is a rank-2 tensor field on $M$ and $R(Y, Z)$ is the curvature operator (Crampin and Pirani, 1986):

$$[R(Y, Z)X]^a = R^a{}_{bcd} X^b Y^c Z^d$$

for all $X$. By (28) and (29), the ATLs form a Lie algebra. The LTLs are a subalgebra (but not an ideal).

The class of ATLs includes all of the lifts previously defined in Section 2, as we shall now show.

**Examples of LTLs**

Before limiting ourselves to the linear case, we regain the vertical lift of a vector field. In order to get $\hat{Z}$, we choose $Y^a = 0$, $A^a{}_b = 0$, $k^a = Z^a$ in (25). Thus,

$$\hat{Z} = 0^{(0,Z)} \quad \text{(30)}$$

Now we limit ourselves to the class of LTLs given by $k^a = 0$: $Y^{(A)} = Y^{(A,0)}$.

By (25a), if $Y = 0$, we regain the vertical lift (18) of the rank-2 tensor field $A$ along with its special case the Euler vector field (19):

$$0^{(A)} = A^a{}_b p^b V_a = \hat{A} \quad \text{(31a)}$$

$$0^{(\delta)} = p^a V_a = \Delta \quad \text{(31b)}$$

By (24), $0^{(A)}$ generates a $GL(n)$ transformation on each fiber:

$$p^a \to p'^a = (e^{\varepsilon A})^a{}_b p^b$$

Thus on each fiber $T_x M$, $A^a{}_b(x)$ is an element of the Lie algebra $gl(n)$. By restricting $A^a{}_b(x)$ to a particular Lie algebra $g$, we see that $0^{(A)}$ generates *gauge transformations* of the corresponding Lie group $G$.

In order to regain the horizontal lift (14) of a vector field, we require that the transport rule $\Lambda$ be *parallel transport* along $Y$:

$$\nabla_Y u = 0$$

for all $u$. By (27) this implies $A = 0$. Then, by (25a) and (14),

$$\bar{Y} = Y^{(0)} \quad \text{(32)}$$

Thus, the horizonal lift $\bar{Y}$ is given clear geometric interpretation as the lifted vector field along whose integral curves arbitrary vectors are parallel transported.

Now we show that when the transport rule $\Lambda$ is chosen to be *Lie transport* ("dragging along"), we regain the complete lift (16). Lie transport along $Y$ implies

$$\mathscr{L}_Y u \equiv \nabla_Y u - \nabla_u Y = 0$$

for all $u$, which by (27) implies

$$A^a{}_b u^b = u^b \nabla_b Y^a$$

for all $u^a$. Thus, $A = \nabla Y$, and by (25a)

$$\tilde{Y} = Y^{(\nabla Y)} \tag{33}$$

Now by (26a) and (26b), the integral curves of $\tilde{Y}$ are given by

$$\frac{dx^a}{d\sigma} = Y^a, \qquad \frac{dp^a}{d\sigma} = Y^a{}_{,b} p^b$$

Thus

$$p'^a = \frac{\partial x'^a}{\partial x^b} p^b$$

and so the flow of $\tilde{Y}$ is

$$(x, p) \to (\phi_\varepsilon x, \phi_{\varepsilon *} p)$$

where $\phi_\varepsilon$ is the flow of $Y$. This property of $\tilde{Y}$ is clearly consistent with the geometric interpretation in terms of Lie transport. It also means that the natural way of defining invariance of a tensor field $\Sigma$ on $TM$ under the point transformations generated by $Y$ on $M$ is

$$\mathscr{L}_{\tilde{Y}} \Sigma = 0$$

[See Maartens and Maharaj (1985) for the case where $\Sigma$ is the distribution function in RKT.] Finally, we note that an important difference between the horizontal and complete lifts is obscured by this unified approach. The occurrence of $Y^a{}_{,b}$ in $\tilde{Y}$ but not in $\bar{Y}$ reflects the fact that Lie transport, unlike parallel transport, is only defined for vector fields.

Thus, we are able to regain in a unified and geometric way the standard lifts of vectors and rank-2 tensors via the concept of ATLs. Using the general Lie bracket relation (29), we can also regain the Lie brackets (Yano and Ishihara, 1973; Crampin, 1983; Prince and Crampin, 1984; Crampin and

Pirani, 1986) among the three standard vector lifts:

$$[\bar{Y}, \bar{Z}] = \overline{[Y, Z]} - R(\widehat{Y, Z})$$
$$[\bar{Y}, \hat{Z}] = \widehat{\nabla_Y Z}$$
$$[\bar{Y}, \tilde{Z}] = \overline{[Y, Z]} + S(\widehat{Y, Z})$$
$$[\hat{Y}, \hat{Z}] = 0$$
$$[\hat{Y}, \tilde{Z}] = \widehat{[Y, Z]}$$
$$[\tilde{Y}, \tilde{Z}] = \widetilde{[Y, Z]}$$

where the operator $S(Y, Z)$ is defined by

$$[S(Y, Z)X]^a = (\mathscr{L}_Z \Gamma^a{}_{cb}) Y^c X^b$$

for all $X^a$. Note that on a curved manifold the sets of vertical and complete lifts each form a Lie algebra, but the horizontal lifts do not. By (29), the vertical lifts form an ideal in the algebra of ATLs, but the complete lifts do not.

We now show that the LTLs also include the matter symmetry vector fields of RKT. Berezdivin and Sachs (1973) define a matter symmetry as a vector field on $TM$ that leaves the distribution function $f$ unchanged. This vector field connects points in $TM$ where the distribution of matter is the same. Geometrically, this implies that an observer at $x$ with local Lorentz frame $F$ will measure $f$ on the tangent fiber $T_x M$ to be the same as an observer at $x'$ with Lorentz frame $F'$ measuring $f'$ on $T_{x'} M$. Thus, matter symmetries arise in the class of LTLs out of the requirement that the transport rule $\Lambda$ be *Lorentz transport* along $Y$. Thus, any vector transforms according to a representation of the Lorentz group $SO(1, 3)$ along $Y$. Given an orthonormal tetrad $\{E_a\}$, we have

$$E_a \cdot E_b = \eta_{ab} \equiv \operatorname{diag}(-1, 1, 1, 1)$$

and the tetrad components of the connection are

$$\Gamma_{abc} = E_a \cdot \nabla_c E_b = -E_b \cdot \nabla_c E_a = -\Gamma_{bac} \tag{34}$$

Now the tetrad components of any vector transform as

$$u'^a = \Lambda^a(u, x; \varepsilon) = \Omega^a{}_b(x; \varepsilon) u^b$$

where $\Omega \in SO(1, 3)$. Thus $\Omega$ preserves $\eta$:

$$\Omega^a{}_c \eta_{ab} \Omega^b{}_d = \eta_{cd}$$

Differentiating and noting that $\Omega^a{}_b(x;0)=\delta^a{}_b$, we get

$$\omega_{(ab)}=0$$

where $\omega$ is defined by (25c). Thus, by (25b) and (34), we have

$$A_{(ab)}=0 \tag{35}$$

which is the condition for $Y^{(A)}$ to be a matter symmetry (or "Lorentz lift"). [Berezdivin and Sachs (1973) derive the condition (35) by requiring that the matter symmetry leaves $m^2=-p^ap_a$ invariant.] The matter symmetries form a Lie algebra, since by (29c), $C$ is skew if $A$ and $B$ are.

Lorentz transport generalizes to (semi-) Riemannian manifolds of any dimension $n$ and signature $n-p$, where vectors transform according to a representation of $SO(p, n-p)$ under $\Lambda$. Then in an orthonormal basis $\{E_a\}$

$$E_a\cdot E_b=\mu_{ab}\equiv\text{diag}(\underbrace{-1,\ldots,-1}_{p},\ \underbrace{1,\ldots,1}_{n-p})$$

and $\Omega\in SO(p, n-p)$ preserves $\mu$, leading to skew-symmetry of $A$ as before:

$$\Omega^a{}_c\,\mu_{ab}\Omega^b{}_d=\mu_{cd}\ \Rightarrow\ A_{(ab)}=0$$

We can define a new lift as a special case of the Lorentz lift. This occurs when the transport rule $\Lambda$ is *Fermi–Walker transport* along $Y$. Thus, the Fermi–Walker derivative (Stephani, 1982) of any $u$ is zero:

$$\mathscr{F}_Y u\equiv\nabla_Y u-(u\cdot\nabla_Y Y)+(u\cdot Y)\nabla_Y Y=0$$

By (27) this implies

$$A^a{}_b=Y^aY_{b;c}Y^c-Y^a{}_{;c}Y^cY_b$$

so that

$$A_{ab}=(Y\wedge\nabla_Y Y)_{ab}=-A_{ba}$$

Then by (25a) we can define the Fermi–Walker lift $Y^*$ of $Y$ as the LTL

$$Y^*=Y^{(Y\wedge\nabla_Y Y)} \tag{36}$$

Iwai's lift (17) arises as the LTL which is the lift of *conformal Lie transport*. However, Iwai defines his lift for $Y$ a projective collineation or conformal Killing vector, whereas the class of ATLs generalizes this to any $Y$. If the transport rule $\Lambda$ is conformal Lie transport, then

$$\mathscr{L}_Y u=-2\psi u$$

for all $u$ and some scalar field $\psi(x)$ on $M$. Then by (27)

$$A^a{}_b=Y^a{}_{;b}-2\psi\delta^a{}_b$$

and by (25a) Iwai's lift generalized to any $Y$ is the LTL

$$Y^{\dagger} = Y^{(\nabla Y - 2\psi\delta)} \tag{37}$$

The generalized Iwai lifts form a Lie algebra, since (37) and (29) imply

$$[Y^{\dagger}, Z^{\dagger}] = [Y, Z]^{\dagger} \qquad \text{where} \quad \psi_{[Y,Z]} = \mathscr{L}_Y \psi_Z - \mathscr{L}_Z \psi_Y$$

with the aid of the Ricci identity and the symmetries of $R$. [This generalizes Iwai's (1977) result to the case of arbitrary $Y$, $Z$.]

We have seen that not only the standard lifts $\hat{Y}$, $\tilde{Y}$, $\bar{Y}$, but also the generalizations due to Iwai and to Berezdivin and Sachs, are all contained within the class of ATLs. Furthermore, all are given clear geometrical interpretations through the concept of ATLs.

## 4. DYNAMICAL AND MATTER SYMMETRIES

In searching for a dynamical symmetry $\Sigma$ [obeying the condition (21)], it is usually assumed that $\Sigma$ arises purely from a vector field on the base manifold $M$—for example, $\Sigma = \tilde{Y}$ or $Y^{\dagger}$. The ATLs open up the possibility of generalizing dynamical symmetries to the case where not only a vector field, but also a transport law for tangent vectors, is used to generate transformations of the dynamical trajectories. In the case of affine transport laws, this means looking at the ATLs. Unfortunately, as we shall show, the dynamical symmetry condition reduces the ATL to a vector lift—in fact, to $Y^{\dagger}$. At least this gives a foundation to the *ad hoc* ansatz of Iwai.

We examine now the conditions under which an ATL is a dynamical symmetry. By (21) this gives

$$[Y^{(A,k)}, \Gamma] = -\psi\Gamma \tag{38a}$$

where

$$\Gamma = p^a H_a + eF^a{}_b p^b V_a + f^a V_a \tag{38b}$$

incorporates the dynamical fields (10a), (10b), and (13). Then (25a) and (38) imply [using (7)–(9)]

$$(A^a{}_b - Y^a{}_{;b})p^b + k^a = -\psi p^a \tag{39a}$$

$$\begin{aligned}(eF^a{}_{b;c}Y^c + [F, A]^a{}_b + R^a{}_{bcd}p^c Y^d - A^a{}_{b;c}p^c - k^a{}_{;b})p^b \\ + eF^a{}_b k^b + f^a{}_{;b}Y^b - A^a{}_b f^b = -e\psi F^a{}_b p^b - \psi f^a\end{aligned} \tag{39b}$$

From (39a), $k^a = 0$, which is the restriction to the class of LTLs. A further implication of (39a) is that

$$A_{ab} = Y_{a;b} - \psi g_{ab} \tag{40}$$

From (40) it is clear that $\psi$ is restricted to $\psi = \psi(x)$. Using (40) in (39b), we get

$$\mathscr{L}_Y \Gamma^a{}_{bc} \equiv Y^a{}_{;bc} - R^a{}_{bcd} Y^d = \delta^a{}_{(b} \psi_{,c)} \tag{41a}$$

$$\mathscr{L}_Y F^a{}_b = -\psi F^a{}_b \tag{41b}$$

$$\mathscr{L}_Y f^a = -2\psi f^a \tag{41c}$$

By (41a), $Y$ is a projective collineation vector (Crampin, 1983; Prince and Crampin, 1984), and together with (40), this means that the ATL is reduced to Iwai's projective lift (37) (although note that Iwai did not consider charged particles):

$$[Y^{(A,k)}, \Gamma] = -\psi \Gamma \;\Rightarrow\; Y^{(A,k)} = Y^{(\nabla Y - \psi\delta, 0)} = Y^\dagger$$

Thus we see that any affinely based dynamical symmetry arises from a projective collineation vector. Furthermore, the ansatz introduced by Iwai in fact arises as the condition for an ATL to be a dynamical symmetry. Any attempt to generalize Iwai's ansatz would require a fully nonlinear transport rule $\Lambda$.

The conformal transformation of $f^a$ under $Y$ in (41c) was also given by Iwai. Our generalization to include relativistic charged particle motion produces (41b), which is a conformal transformation of the electromagnetic field tensor $F^a{}_b$ under $Y$. [In fact, results similar to (41a)–(41c) appear to have first been found not by Iwai, but by Katzin and Levine (1974), without reference to the tangent bundle and liftings. Katzin and Levine, like Iwai, considered only symmetries generated by a base vector field.]

Finally, we note that in RKT, if a matter symmetry is also a dynamical symmetry (i.e., an invariance mapping of trajectories as well as of the distribution function), then by (35) and (40) we have

$$Y_{(a;b)} = \psi g_{ab}$$

which, together with (41a), implies

$$\psi_{,a} = 0$$

Thus, a matter symmetry is simultaneously a dynamical symmetry only if $Y$ is a homothetic Killing vector and

$$A = dY$$

Furthermore, in the case of charged particles, $F^a{}_b$ must map homothetically under $Y$, by (41b). These are stringent restrictions, and show that matter symmetries (distribution-based) are in general very different from dynamical symmetries (trajectory-based).

## 5. CONCLUSION

By generalizing the concept of lifting point transformations to include tangent vector transport, we have defined the class of ATLs on the tangent bundle. The ATLs include all previous lifts, thus unifying many results into a single framework, with clear geometric interpretations. The generalization introduced by the ATL concept includes in particular the matter symmetries of RKT and the lifts introduced *ad hoc* by Iwai. The projective lift of Iwai is shown to be the unique ATL which is a dynamical symmetry on (semi-) Riemannian manifolds. The matter symmetries provide a very different concept of invariance, coinciding with dynamical symmetries only in the special case that $Y$ is homothetic and $A = dY$. The study of matter symmetries in their own right, and in particular their relation to geometric symmetries, is taken up in the second paper (Maartens and Taylor, in preparation) of this research program.

Applications of the ATL formalism beyond RKT are possible. It may also be applicable to the study of symmetries in gauge field theories, since $Y^{(A)}$ generates gauge transformations along $Y$ if $A$ is in the gauge Lie algebra at each point. The formalism could also be generalized to other fiber bundles. For example, an ATL on the $\binom{r}{s}$ tensor bundle $T^r_sM$ arises when $\Lambda$ transforms $\binom{r}{s}$ tensors along $Y$. With modifications, the formalism would also carry through to the tangent bundle of a manifold with torsion.

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